Math 246B Lecture 19 Notes

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1 Applications of Hadamard Factorization and Properties of the Γ-Function

1.1 Minimum modulus theorem and range of entire functions of finite order

Last time, we proved the Hadamard factorization for entire functions of finite order:

$$f(z) = e^{g(z)} z^p \prod_{k-1}^{\infty} E_m(z/a_k),$$

where (a_k) are the zeros of f such that $0 < |a_1| \leq |a_2| \leq \cdots$, p is the order of the zeros at 0, $m \leq \rho < m+1$, and g is a polynomial of degree $\leq \rho$. We have for all $s \in (\rho, m+1)$ there exists some $C > 0$ such that

$$\left| \prod E_m(z/a_k) \right| \geq e^{-C|z|^s}, \qquad z \in \mathbb{C} \setminus \bigcup D(a_k, 1/|a_k|^{m+1}).$$

Our analysis of this gives us the following facts:

Corollary 1.1 (minimum modulus theorem)**.** *For every $\varepsilon > 0$, there exists an $R > 0$ such that*

$$|f(z)| \geq e^{-|z|^{\rho+\varepsilon}}, \qquad |z| \geq R, \quad z \in \mathbb{C} \setminus \bigcup D(a_k, 1/|a_k|^{m+1}).$$

Corollary 1.2. *Let f be entire of finite order $\rho \notin \mathbb{N}$. Then f assumes every complex value infinitely many times.*

Proof. For any $w \in \mathbb{C}$, $f, f - w$ are entire of the same order, so it suffices to show that f has infinitely many zeros. If f has only finitely many zeros, then the Hadamard factorization gives $f(z) = p(z)e^{g(z)}$, where p, g are polynomials. The order of such a function is the degree of g, which is an integer. $\square$

1.2 Factorization of sine

Example 1.1. Let $f(z) = \sin(\pi z)$. This is entire of order 1, and $f^{-1}(\{0\}) = \mathbb{Z}$. Write $\mathbb{Z} \setminus \{0\}$ as $\{a_k : k = 1, 2, \dots\}$ with $a_{2j} = -j$ for $j \geq 1$ and $a_{2j+1} = j+1$, for $j \geq 0$. We can write

$$\begin{aligned}
\sin(\pi z) &= e^{g(z)} z \prod_{k=1}^{\infty} E_1(z/a_k) \\
&= e^{g(z)} z \prod_{k=1}^{\infty} (1 - z/a_k) e^{z/a_k} \\
&= e^{g(z)} z \prod_{j=1}^{\infty} (1 + z/j) e^{-z/j} \prod_{j=0}^{\infty} (1 - z/(j+1)) e^{z/(j+1)} \\
&= e^{g(z)} z \prod_{j=1}^{\infty} (1 + z^2/j^2)
\end{aligned}$$

e^g is even, and g is a polynomial of degree ≤ 1. So $g(z) = g(= z) + 2\pi k i$ for some $k \in \mathbb{Z}$. If $g(z) = \alpha z + \beta$, then $\alpha = 0$.

$$= e^{\beta} z \prod_{j=1}^{\infty} (1 + z^2/j^2).$$

To find β, differentiate and take $z = 0$ to get $\pi = e^{\beta}$. This gives us the classical factorization formula:

$$\sin(\pi z) = \pi z \prod_{j=1}^{\infty} (1 - z^2/j^2).$$

1.3 The Γ-function

Definition 1.1. The **Γ-function** is defined by

$$\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} \, dt, \qquad \mathrm{Re}(a) > 0.$$

The integral converges locally uniformly in $\mathrm{Re}(a) > 0$ and defines a holomorphic function in this region. We have

$$\Gamma(a+1) = \lim_{\substack{\varepsilon \to 0^+ \\ R \to \infty}} \int_{\varepsilon}^{R} e^{-t} t^a \, dt = \lim_{\substack{\varepsilon \to 0^+ \\ R \to \infty}} \left(-t^a e^{-t} \Big|_{\varepsilon}^{R} + \int_{\varepsilon}^{R} a t^{a-1} e^{-t} \, dt \right) = a\Gamma(a),$$

when $\mathrm{Re}(a) > 0$. In particular, since $\Gamma(1) = 1$, we have

$$\Gamma(n) = (n-1)!, \qquad n \geq 1.$$

Proposition 1.1. *The* Γ*-function has a meromorphic continuation to* $\mathbb{C}$ *with simple poles at the nonpositive integers* $\{0, -1, -2, \dots\}$*. The residue at* $-N$ *is* $(-1)^N/N!$*.*

Proof. For $N \in \mathbb{N}$ with $N > 0$, write

$$\begin{aligned}
\Gamma(a+N+1) &= (a+N)\Gamma(a+N) \\
&= (a+N)(a+N-1)\Gamma(a+N-1) \\
&= \cdots \\
&= (a+N)\cdots(a+1)a\Gamma(a).
\end{aligned}$$

So we can write

$$\Gamma(a) = \frac{\Gamma(a+N+1)}{(a+N)\cdots(a+1)a}.$$

The right hand side is meromorphic in $\mathrm{Re}(a) > -N-1$. Thus, Γ extends meromorphically to all of $\mathbb{C}$ with the poles $\{0, -1, -2, \dots\}$. Compute

$$\mathrm{Res}(\Gamma, -N) = \lim_{a \to -N}(a+N)\Gamma(a) = \frac{(-1)^N}{N!}$$ □

Remark 1.1. We have $\Gamma(a+1) = a\Gamma(a)$ for $a \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$.

We want to apply Hadamard factorization to Γ, but it is not entire. However, $1/\Gamma$ is entire. We will use the following property of the Γ function:

Proposition 1.2 (reflection identity)**.** *For* $a \in \mathbb{C} \setminus \mathbb{Z}$*,*

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

Proof. It suffices to show the identity when $0 < \mathrm{Re}(a) < 1$. Write

$$\Gamma(1-a) = \int_0^\infty e^{-x}x^-a\,dx \overset{x=ty}{==} t\int_0^\infty e^{-ty}(ty)^{-a}\,dy.$$

so we may write

$$\begin{aligned}
\Gamma(a)\Gamma(1-a) &= \int_0^\infty e^{-t}t^{a-1}t\left(\int_0^\infty e^{-ty}(ty)^{-a}\,dy\right)dt \quad = \iint_{t\geq 0, y\geq 0} e^{-t(1+y)}y^{-a}\,dy\,dt \\
&= \int_0^\infty \frac{y^{-a}}{1+y}\,dy \\
&= \frac{\pi}{\sin(\pi a)}.
\end{aligned}$$

To show the last equality apply the residue theorem to

$$f(z) = \frac{z^{b-a}}{1+z}$$

with $0 < b < 1$ and $0 < \arg(z) < 2\pi$, using a "keyhole contour." We get

$$\int_\gamma f(z),\ dz \to (1 - e^{2\pi i(b-1)}) \textstyle\int_0^\infty \dfrac{x^{b-1}}{1+x}\, dx,$$

where the left hand side equals $2\pi i(-1)^{b-1}$. $\square$

Next time, we will show that $1/\Gamma$ is entire of order 1.